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COHERENT DETECTION ON
PULSED RADARS

by

Anthony Kerdock

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ABSTRACT

In this report, coherent detection on a pulsed radar is discussed from a statistical decision theory viewpoint. The Neyman Pearson test is applied to two cases; first, where it is desired to detect targets moving with one particular radial velocity, and second, where it is desired to detect targets at all velocities equally well. In the first case it is shown that, in a sense, an ideal integrator is achieved; i. e., the results are exactly the same as if all the power reflected from the target were received in one pulse, rather than many. A mechanization for the second case is given. The statistics for a suboptimum integrator which approximates the Neyman Pearson test in a simpler form are derived. The performance of this integrator is compared with that for the ideal noncoherent integrator.

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Chapter 1

INTRODUCTION

In order to improve target detection capabilities of pulsed radars for target returns with small signal to noise ratios, a device which combines the returns from several pulse transmissions is often used. This device, known as the integrator, is generally of the noncoherent type, i.e., it sums the radar returns after detection of the I. F. output, and makes no use of phase information. The noncoherent integrator has been exhaustively treated by Marcum[1] in his "Statistical Theory of Target Detection by Pulsed Radar." The results of Marcum's paper imply that a detection scheme which makes use of both amplitude and phase of the I. F. signal returns should be able to provide increased signal detectability.

In this report, the Neyman Pearson detector, which uses all the available information in the amplitude and phase of the signal return is derived and mechanized. The Neyman Pearson detector is optimum in the sense that for a given false alarm probability, (the probability that noise is mistaken for a target), it gives the greatest probability of detection for a target. For false alarm probabilities smaller than 10^{-6}, a simpler suboptimum coherent detector which gives nearly optimum performance is developed. The characteristics of this coherent detector are examined and compared with those for the noncoherent integrator treated by Marcum.[1]

Chapter 2

STATISTICS OF THE RADAR RETURN

This report will be confined to a discussion of pulsed radars. A short pulse of duration τ is transmitted every T seconds. Typically, τ/T may be of the order of 10^{-3}. The returned R.F. echo from a target is usually heterodyned down to some lower I.F. frequency. This I.F. signal has both amplitude and phase information, although special processing may be required to utilize the phase information.

A typical schematized transmitting and receiving system for a radar whose transmitter signal is derived from a crystal oscillator and frequency multiplier chain is shown in Fig. 1. The I.F. output has a fixed phase shift from the reference oscillator of $\theta_2 + \theta_3 + (2d/\lambda)2\pi$, where θ_2 and θ_3 are fixed phase shifts inherent in the system, and $2d/\lambda$ is the round trip distance to the target in wavelengths of the transmitted frequency. If the target is stationary with respect to the radar, then the phase of the echo pulse does not change from return to return. If the target has a radial component of velocity with respect to the radar, then the change of phase of the echo from pulse to pulse is $\Delta\phi = 2(dd/dt)/\lambda = 4\pi vT/\lambda$; where v is the radial velocity of the target with respect to the radar. Essentially, the frequency returned is modified by the doppler frequency, $fd = 2v/\lambda$, which is sampled every T seconds, resulting in a phase change of $\Delta\phi = 4\pi vT/\lambda$.

Phase information may also be obtained on a radar that has a pulsed oscillator such as a magnetron for a transmitter, although more, and more critical circuitry is required. A system designed to preserve phase information is diagrammed in Fig. 2. The I.F. obtained after heterodyning the returned echo with the STALO, (a highly stable local oscillator), has a random phase term θ_r, in it, which appears because the transmitter fires at a random phase from pulse to pulse. This random phase term, θ_r, can be removed however, by heterodyning the I.F. with an oscillator which is locked in phase with the transmitter after each transmission. This type of oscillator, called a COHO, (coherent oscillator), is widely used for M.T.I. (moving target indication), applications. Extreme stability is required of the STALO and COHO in this system to preserve the phase information. With the requisite stability, the same kind of output as in the crystal controlled radar is obtained.

The signal from the n^{th} transmission of the radar is seen to be,

$$V_n(t) = A_n \cos(\omega_c t + \delta + n\psi), \tag{1}$$

where A_n is the amplitude of the signal return, ω_c the I.F. center frequency, δ a random initial phase dependent upon the exact round trip distance of the target from the radar, and ψ a phase precession term dependent on the radial velocity. If the radial velocity of the target is in m.p.h., and the wavelength of the transmitted R.F. is in centimeters, the phase shift from pulse to pulse is $\psi = (89V/\lambda)2\pi T$. The phase shift during the pulse due to the doppler frequency in the echo return is quite small for most radars, and is neglected since it does not affect the results which follow.

The signal is accompanied by a narrow band gaussian noise process, whose spectrum is shaped by the band pass characteristics of the I.F. amplifier through which it passes. A sample function of this random process may be represented(2) as,

$$V(t) = r(t) \cos\left[\omega_c t + \theta(t)\right] \tag{2}$$

where $r(t)$ is the envelope process, and $\theta(t)$ is the phase process, both of which vary slowly with respect to ω_c. Expanding

$$\begin{aligned} V(t) &= r(t) \cos\left[\omega_c(t) + \theta(t)\right] \\ &= r(t) \cos\theta(t) \cos\omega_c t - r(t) \sin\theta(t) \sin\omega_c t \\ &= x(t) \cos\omega_c t - y(t) \sin\omega_c t \end{aligned} \tag{3}$$

x and y are statistically independent gaussian random variables having a joint distribution

$$p(x, y) = \frac{e^{-\frac{x^2+y^2}{2\sigma^2}}}{2\pi\sigma^2} \tag{4}$$

where σ^2 is the variance, or physically, is the noise power into a one ohm resistor.

The noise has been shaped by an I.F. bandwidth matched to the transmitted pulse width τ. Thus two samples of noise taken at intervals much greater than τ apart show little corellation. Since T is usually greater than 100τ, noise samples taken a transmission period apart are essentially independent.

Thus the joint probability distribution for noise, at a given range, from N transmission periods is

$$p(x_1, x_2, \dots, x_N, y_1, y_2, \dots, y_N) = \frac{e^{-\sum_{n=1}^{N}(x_n^2+y_n^2/2\sigma^2)}}{(2\pi\sigma^2)^N} \quad (5)$$

The signal from the n^{th} transmission was seen to be

$$V_n(t) = A_n \cos(\omega_c t + \delta + n\psi)$$
$$= A_n \cos(\delta + n\psi) \cos \omega_c t - A_n \sin(\delta + n\psi) \sin \omega_c t$$

The x component is thus $A_n \cos(\delta + n\psi)$, and the y component of the signal is $A_n \sin(\delta + n\psi)$. The probability distribution for signal plus noise is thus,

$$p_{s+N}(x_n, y_n) = \frac{e^{-(x_n - A_n\cos(\delta+n\psi))^2 - (y_n - A_n\sin(\delta+n\psi))^2/2\sigma^2}}{2\pi\sigma^2} \quad (6)$$

The probability density distribution for the return from N transmission periods is

$$p_{s+N}(x_1, x_2 \dots x_N, y_1, y_2 \dots y_N) = \frac{e^{-\sum_{n=1}^{N}(x_n - A_n\cos(\delta+n\psi))^2 + (y_n - A_n\sin(\delta+n\psi))^2/2\sigma^2}}{(2\pi\sigma^2)^N} \quad (7)$$

It will be found convenient later to work with the envelope and phase distributions rather than with the x and y distributions. The probability density distribution for sine wave plus noise in terms of envelope and phase is well known, and is given by Davenport and Root "Random Signals and Noise",[2] and others.

$$p_{s+N}(r_n, \theta_n) = \frac{r_n}{2\pi\sigma^2} e^{-(r_n^2+A_n^2)/2\sigma^2} e^{-r_n A_n \cos(\theta_n - \delta - n\psi)/\sigma^2} \quad (8)$$

where $r_n = \sqrt{x_n^2+y_n^2}$, and $\theta_n = \tan^{-1} y_n/x_n$.

For noise alone, i.e., $A_n = 0$,

$$p_N(r_n, \theta_n) = \frac{r_n}{2\pi\sigma^2} e^{-r_n^2/2\sigma^2} \tag{9}$$

which is the well known Rayleigh distribution times $1/2\pi$, the probability distribution of phase. The joint distribution from N returns is thus for signal plus noise

$$p_{s+N}(r_1, r_2, \ldots, r_N, \theta_1, \theta_2, \ldots, \theta_N) =$$

$$= \frac{\prod_{n=1}^{N} r_n}{(2\pi\sigma^2)^N} e^{-\sum_{n=1}^{N} r_n^2 + A_n^2/2\sigma^2} e^{1/\sigma^2 \sum_{n=1}^{N} r_n A_n \cos(\theta_n - \delta - n\psi)} \tag{10}$$

For noise alone

$$p_N(r_1, r_2, \ldots, r_N, \theta_1, \theta_2, \ldots \theta_N) = \frac{\prod_{n=1}^{N} r_n}{(2\pi\sigma^2)^N} e^{-\sum_{n=1}^{N} r_n^2/2\sigma^2} \tag{11}$$

Chapter 3

THE NEYMAN PEARSON TEST

A procedure is desired for making a decision as to whether or not a target exists at a given range and azmuth. The decision must be based on the observed data, (the x_n's and y_n's or r_n's and θ_n's). No apriori knowledge of the number of targets in the area can be assumed, except that it will be small as compared with the number of independent noise samples taken. What is generally done in an automatic detection system, is to set the noise threshold so as to obtain a false alarm rate, (rate of error in calling noise spikes targets), of one or less per scan. Since there are approximately T/τ independent noise samples per transmission, and often thousands of transmissions per scan, we should consider probabilities that an individual noise spike will be mistaken for a target of from 10^{-6} to 10^{-10}.

It is desired to find a test which will give the greatest probability of detection for a given false alarm probability. The Neyman Pearson test does exactly this. For a test between two hypotheses, H_0, (called the null hypothesis), and H_1, at a given level, (probability of mistaking H_0 for H_1), it gives a test of maximum power, (probability of choosing H_1 when it is true). The Neyman Pearson test is made by forming the ratio of the probability density distributions for H_1 and H_0, $p_1(y)/p_0(y)$, called the likelihood ratio. The value of the observed parameters are substituted in this expression, and a number is obtained. If this number is greater than some number predetermined by the desired false alarm probability, the hypothesis H_1 is chosen.

In our case, the null hypothesis is that there is only noise, and $p_0 = p_N(r_i, \ldots, r_N, \theta_1, \ldots, \theta_N)$. The hypothesis H_1 is that there is a target present with an initial phase δ, and a phase precession ψ. The likelihood ratio is

$$\frac{p_{\delta\psi}{}_{S+N}\{r_n, \theta_n\}}{p_N\{r_n, \theta_n\}}$$

where p'_{S+N} is given by (10) and p_N is given by (11). The test is then, whether

$$\frac{p_{\delta\psi}{}_{S+N}}{p_N} = e^{-\sum_{n=1}^{N} A_n/2\sigma^2} \cdot e^{\frac{1}{\sigma^2}\sum_{n=1}^{N} r_n A_n \cos(\theta_n - \delta - n\psi)} \geq K \qquad (12)$$

The above test has no practical significance however, since it is only a test of a target with a particular phase δ, and a doppler velocity corresponding to the precession angle ψ, versus noise. Since we have no apriori knowledge that a target will give a return with some particular value of δ, and have no interest in detecting only targets with specific values of this parameter, a more general test must be devised. What is primarily desired, is a test of the composite hypothesis that the target can have any value of δ and ψ , versus noise. The adaptation of a Neyman Pearson test for a composite hypothesis versus a simple hypothesis is given in Appendix A. It is shown there, that if we make the test,

$$\frac{\int_0^{2\pi}\int_0^{2\pi} p_{\delta,\psi\, S+N}(r_1, r_2, \dots r_N, \theta_1, \theta_2, \dots \theta_N)p(\delta)p(\psi)d\delta\, d\psi}{p_N(r_1, r_2, \dots r_N, \theta_1, \theta_2, \dots \theta_N)} \geq K \qquad (13)$$

in effect averaging over δ and ψ that we will have the test of composite H_1, versus H_0, of the greatest power for a given level. $p(\delta)$ and $p(\psi)$ are the probability density distributions for δ and ψ respectively.

The target return may have any phase with equal probability, since λ is very small compared to the target distance. Thus

$$\begin{aligned} p(\delta) &= 1/2\pi \quad \text{for} \quad 0 \leq \delta \leq 2\pi \\ &= 0 \quad \text{otherwise} . \end{aligned} \qquad (14)$$

Similarly, with most radars, the phase precession is greater than 2π with a radial velocity of less than 200 knots. Thus for aircraft we may say that

$$\begin{aligned} p(\psi) &= 1/2\pi \quad \text{for} \quad 0 \leq \psi \leq 2\pi \\ &= 0 \quad \text{otherwise.} \end{aligned} \qquad (15)$$

The integral, $\int_0^{2\pi} p_{S+N}(\{r_n, \theta_n\})p(\delta)\, d\delta$

may be readily evaluated.

$$\int_0^{2\pi} p_{S+N}(\{r_n, \theta_n\})p(\delta)d\delta = \qquad (16)$$

$$= \frac{\prod_{n=1}^{N} r_n}{(2\pi\sigma^2)^N} e^{-\sum_{n=1}^{N} \frac{r_n^2 + A_n^2}{2\sigma^2}} \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{1}{\sigma^2}\sum_{n=1}^{N} A_n r_n \cos(\theta_n - n\psi - \delta)} d\delta$$

let $\theta_n - n\psi = \psi_n$

then
$$\sum_{n=1}^{N} A_n r_n \cos(\phi_n - \delta) = \sum_{n=1}^{N} A_n r_n \cos\phi_n \cos\delta + A_n r_n \sin\phi_n \sin\delta$$

$$= \cos\delta \sum_{n=1}^{N} A_n r_n \cos\phi_n + \sin\delta \sum_{n=1}^{N} A_n r_n \sin\phi_n$$

$$= \cos(\delta - \alpha) \sqrt{\left(\sum_{n=1}^{N} A_n r_n \cos\phi_n\right)^2 + \left(\sum_{n=1}^{N} A_n r_n \sin\phi_n\right)^2} \tag{17}$$

but

$$\frac{1}{2\pi}\int_0^{2\pi} e^{z\cos(\delta-\alpha)}\, d\delta = I_0(z) = J_0(iz) \tag{18}$$

where I_0 is the modified Bessel function of the first kind and zero order, and J_0 is the Bessel function of the first kind and zero order. Thus,

$$\int_0^{2\pi} p_{\delta,\psi}\ s+N\left(\{r_n, \theta_n\}\right) p(\delta)\, d\delta = p_\psi\ s+N\left(\{r_n, \theta_n\}\right)$$

$$= \frac{\prod_{n=1}^{N} r_n}{(2\pi\sigma^2)^N}\, e^{-\sum_{n=1}^{N} r_n^2 + A_n^2/2\sigma^2}\, I_0\left(\frac{1}{\sigma^2}\sqrt{\left(\sum_{n=1}^{N} A_n r_n \cos(\theta_n - n\psi)\right)^2 + \left(\sum_{n=1}^{N} A_n r_n \sin(\theta_n - \psi)\right)^2}\right) \tag{19}$$

This is the probability density distribution independent of the parameter δ, but dependent on ψ.

With this probability distribution, we may set up the test of a target moving with given doppler velocity, or a stationary target, versus noise.

$$\frac{p_{\psi}s+N(\{r_n,\theta_n\})}{p_N(\{r_n,\theta_n\})} = e^{-\sum_{n=1}^{N} A_n^2/2\sigma^2} \; I_0\left(\frac{1}{\sigma^2}\sqrt{\left(\sum_{n=1}^{N} A_n r_n \cos(\theta_n - n\psi)\right)^2 + \left(\sum_{n=1}^{N} A_n r_n \sin(\theta_n - n\psi)\right)^2}\right) \tag{20}$$

is the likelihood ratio, $(p_1(y)/p_0(y))$, we must compare with a number K. The factor

$$e^{-\sum_{n=1}^{N} A_n^2/2\sigma^2}$$

is a multiplying factor dependent on the amplitude and distribution of the signal we are trying to detect, and does not involve any of the observed data(r_n's, and θ_n's). This factor may thus be incorporated into the number K giving K'. The function $I_0(1/\sigma^2 R)$ is monotonically increasing with respect to R, for $R > 0$. Thus $I_0(1/\sigma^2 R)$ is greater than K' if and only if R is greater than some other number K''. The test may then be simplified to

$$\sqrt{\left(\sum_{n=1}^{N} A_n r_n \cos(\theta_n - n\psi)\right)^2 + \left(\sum_{n=1}^{N} A_n r_n \sin(\theta_n - n\psi)\right)^2} \geq K'' \tag{21}$$

When the observed θ_n's and r_n's are substituted in the above expression and compared with number K'' which is determined by the desired false alarm probability, we have the Neyman Pearson test for a stationary target, or target with a given doppler velocity, vs. noise. The r_n's of the I.F. process are obtained by linear envelope detection. The θ_n's can be obtained by phase comparison with the oscillator of Figs. 1 or 2 whose output is $\cos \omega_c t$. The detection inequality may then be mechanized with a digital computor or with an analog scheme using delay lines.

The performance of this test is easily evaluated. Since

$$\sum_{n=1}^{N} A_n r_n \cos(\theta_n - n\psi)$$

and

$$\sum_{n=1}^{N} A_n r_n \sin(\theta_n - n\psi)$$

are linear operations on sine θ_n and cos θ_n we may operate on signal and noise separately, and use superposition to combine the results. The signal from the n^{th} return is $A_n \cos(\omega_c t + \delta + n\psi)$. Without loss of generality, we can take δ equal to zero, since the test has been devised to give the same results for all δ. Then when we rotate back $n\psi$ we find that the $r_n \sin(\theta_n - n\psi)$ component vanishes, and

$$\sum_{n=1}^{N} A_n r_n \cos(\theta_n - n\psi) = \sum_{n=1}^{N} A_n^2$$

In effect we are performing a rotation which adds up all the signals in phase, each with a given weighting.

The noise is invariant under phase rotation, and for a noise process with variance σ^2, the process which results from multiplying the noise voltage by A_n has a variance $A_n^2 \sigma^2$. The sum of N independent gaussian noise processes with variance $A_n^2 \sigma^2$ is a gaussian noise process with variance

$$\sum_{n=1}^{N} A_n^2 \sigma^2$$

The test is then equivalent to comparing the envelope of a sine wave plus noise, to a given threshold. (Taking the square root of the sum of the squares of the two orthogonal components is equivalent to envelope detection).

The signal power is

$$\frac{1}{2}\left(\sum_{n=1}^{N} A_n^2\right)^2$$

and the noise power is

$$\sigma^2 \sum_{n=1}^{N} A_n^2$$

The signal to noise ratio is $1/2 \sum_{n=1}^{N} A_n^2 / \sigma^2$. But $1/2 \sum_{n=1}^{N} A_n^2$ is the total power received from the pulses modulated by the antenna beam-shape as the antenna scans by the target. Since the statistics for signal and noise are the same as for one return, we arrive at the identical result as if all the returned power were concentrated into one pulse.

Although we have been discussing only a target return with signal amplitude A_n, the signal return will more generally generally be $C A_n$, where C is any multiplicative constant. The shape of the set of pulse returns is constant, and is determined by the antenna pattern, and the rate at which it scans by a target. The test for target amplitudes $C A_n$ is the same as that for A_n, and since K'' is determined by the false alarm probability for noise alone, this test is said to be a uniformly most powerful test with respect to the amplitude of the signal return.

While the test described above has great theoretical importance, in that it shows that the optimum detector for a target with a given radial velocity gives the same results as if all the energy were concentrated in one pulse, it is of much more practical interest to find the test which is best for all radial velocities, rather than just one. To do this we must evaluate the ratio

$$\frac{\int_0^{2\pi} p_{\psi S+N}(\{r_n, \theta_n\}) p(\psi) d\psi}{p_N(\{r_n, \theta_n\})} = \tag{22}$$

$$= e^{-\sum_{n=1}^{N} \frac{A_n^2}{2\sigma^2}} \cdot \frac{1}{2\pi} \int_0^{2\pi} I_0 \left(\frac{1}{\sigma^2} \sqrt{\left(\sum_{n=1}^{N} A_n r_n \cos(\theta_n - n\psi)\right)^2 + \left(\sum_{n=1}^{N} A_n r_n \sin(\theta_n - n\psi)\right)^2} \right) d\psi$$

Unfortunately, the integral involved is a difficult one to evaluate. Expansion of I_0 into a power series, and integrating term by term does not help because the integrals of the higher order terms, (which are important), become so complex as to be unmanageable. However, the Neyman Pearson test may be mechanized in its integral form. The test will consist of comparing

$$\int_0^{2\pi} I_0 \left(\frac{1}{\sigma^2} \sqrt{\left(\sum_{n=1}^{N} A_n r_n \cos(\theta_n - n\psi)\right)^2 + \left(\sum_{n=1}^{N} A_n r_n \sin(\theta_n - n\psi)\right)^2} \right) d\psi \tag{23}$$

to some constant, K'.

In its present form, the integral may be difficult to mechanize. However, if we make the substitution,

$$n\psi = (N\psi - (N-n)\psi) \quad ,$$

a little trigonometric manipulation shows,

$$\sqrt{\left(\sum_{n=1}^{N} A_n r_n \cos(\theta_n - n\psi)\right)^2 + \left(\sum_{n=1}^{N} A_n r_n \sin(\theta_n - n\psi)\right)^2} =$$

$$= \sqrt{\left(\sum_{n=1}^{N} A_n r_n \cos(\theta_n + (N-n)\psi)\right)^2 + \left(\sum_{n=1}^{N} A_n r_n \sin(\theta_n + (N-n)\psi)\right)^2} \qquad (24)$$

for every ψ. This last expression is more easily mechanized than the first.

Fig. 3 shows a theoretically possible mechanization of the expression (23) by means of the identity (24). The I.F. output from the systems shown in Figs. 1 or 2 is the input to this system, and is processed before detection. Each of the N-1 delay lines has delay T, the pulse repetition period of the radar. Thus the N signal returns from a target, multiplied by the weighting factors A_n are simultaneously available at the input to the summer. Preceding the delay lines are voltage variable phase shifters, which are programmed by means of a sawtooth voltage to synchronously change their phase by 2π over a time equal or less than a pulse width τ. Voltage variable delay lines which might be used for this purpose are available. The n^{th} return goes through N-n phase shifters, and has an I.F. cosine component of $A_n r_n \cos(\theta_n + (N-n)\psi)$, at the input to the summer. The summer adds all the N I.F. inputs, so that the output I.F. of the summer has an I.F. cosine component of

$$\sum_{n=1}^{N} A_n r_n \cos(\theta_n + (N-n)\psi),$$

and a sine component of $\sum_{n=1}^{N} A_n {}_n \sin(\theta_n + (N-n)\psi)$.

Linear envelope detection of the I.F. is equivalent to taking the square root of the sum of the squares. Therefore the output voltage of the linear envelope detector is

$$R(\psi) = \sqrt{\left(\sum_{n=1}^{N} A_n r_n \cos(\theta_n + (N-n)\psi)\right)^2 + \left(\sum_{n=1}^{N} A_n r_n \sin(\theta_n + (N-n)\psi)\right)^2}$$

This is equal to

$$\sqrt{\left(\sum_{n=1}^{N} A_n\, r_n \cos(\theta_n - n\psi)\right)^2 + \left(\sum_{n=1}^{N} A_n\, r_n \sin(\theta_n - n\psi)\right)^2}$$

by Eq. (24). The envelope detector is followd by a zero memory nonlinear transfer function, (which can be mechanized using diodes), $I_0(R(\psi)/\sigma^2)$. Finally there is a boxcar detector (a diode feeding a capacitor), which integrates $I_0(R(\psi)/\sigma^2)$ over the cycle of ψ which is 2π, and is "dumped", or discharged, at the end of each cycle by a trigger developed from the trailing edge of the sawtooth. The output of the boxcar detector is then put through a threshold circuit which passes as targets only those outputs of the boxcar detector greater than K'.

It is thus seen that the system of Fig. 3 explicitly mechanizes the test given by expression (23). This system could also be used to detect targets of only a given doppler velocity by keeping the phase shifters fixed at the proper phase, rather than sweeping them over 2π. The nonlinear processor, $I_0(R/\sigma^2)$, and the boxcar detector are not really required in this case, (although leaving them in will not affect performance), and the output of the linear detector can be sent directly to the threshold circuit.

It must be emphasized that the circuit of Fig. 3 is only a theoretical mechanization of the Neyman Pearson test, and practical difficulties such as maintaining the delay of N delay lines identical to each other to within a small fraction of $1/f_c$, where f_c is the I.F. carrier frequency, would probably make construction of the detector by this scheme impossible. A digital scheme using the r_n's and θ_n's directly may be feasible.

Chapter 4

PERFORMANCE OF A COHERENT INTEGRATOR

It is desired to evaluate the signal to noise ratio required for a given probability of detection, P_d , with a given probability of false alarm, P_n. For purposes of simplicity, and so that we can compare the results for a coherent integrator with those of Marcum' for the noncoherent integrator, we shall assume that the signal return consists of N equal amplitude pulses. Then

$$\sqrt{\left(\sum_{n=1}^{N} A_n r_n \cos(\theta_n - n\psi)\right)^2 + \left(\sum_{n=1}^{N} A_n r_n \sin(\theta_n - n\psi)\right)^2} = \tag{25}$$

$$= A\sqrt{\left(\sum_{n=1}^{N} r_n \cos(\theta_n - n\psi)\right)^2 + \left(\sum_{n=1}^{N} r_n \sin(\theta_n - n\psi)\right)^2}$$

In processing this signal we would make all the A_n's of Fig. 3 equal to one, and change the nonlinear eleme nt from $I_0(R/\sigma^2)$ to $I_0(A\,R/\sigma^2)$, where now the signal R is

$$R = \sqrt{\left(\sum_{n=1}^{N} r_n \cos(\theta_n - n\psi)\right)^2 + \left(\sum_{n=1}^{N} r_n \sin(\theta_n - n\psi)\right)^2} \tag{26}$$

It is seen that the test depends on the amplitude of the signal we are trying to detect. If the test is set up for a particular value of A and σ^2 in the nonlinear element $I_0(AR/\sigma^2)$, then the test will be most powerful only for that A, and will be less sensitive for all other amplitudes of signal return. Thus we do not have a uniformly most powerful test with respect to A. What would be done in practice, would be to set up $I_0(AR/\sigma^2)$ for the marginal signal, so as to get greatest sensitivity for it. Signals larger than marginal are easily detected anyway.

It is instructive to investigate the value of $I_0(AR/\sigma^2)$. If we are interested in a probability of detection, P_d, of about .50, with a false alarm probability, P_n, of the order of 10^{-6} then $NA^2/2\sigma^2 \approx 15$. $R_{peak} \approx NA$. Thus, $AR/\sigma^2 \approx 30$. For values of x greater than ten, $I_0(x) \simeq e^x/\sqrt{2\pi x}$.(2) Thus if AR/σ^2 were to change from 30 to 31, the output of the nonlinear device would be multiplied by approximately e. It is thus seen that

$$\int_0^{2\pi} I_0\left(\frac{AR(\psi)}{\sigma^2}\right) d\psi$$

is determined primarily by the peaks of $AR(\psi)/\sigma^2$, since these are very heavily weighted.

We should therefore expect that a device which detects the peaks of $R(\psi)$, should give a test nearly as powerful as that illustrated in Fig. 3 for $P_n \lessapprox 10^{-6}$ and $P_d \gtrapprox .50$. Furthermore, this test is uniform, (although not most powerful), with respect to A. This test is mentioned by Reed, Kelly, and Root in "Detection of Radar Echoes in Noise",[3] although this performance of the test is not evaluated. It is this test which will be investigated here, rather than the test of Fig. 3, because the calculations for P_n and P_d are much simpler.

We must first investigate the statistics of $R(\psi)$. Let

$$X(\psi) = \sum_{n=1}^{N} r_n \cos(\theta_n - n\psi) \tag{27}$$

$$Y(\psi) = \sum_{n=1}^{N} r_n \sin(\theta_n - n\psi) \tag{28}$$

Then

$$R(\psi) = \sqrt{X(\psi)^2 + Y(\psi)^2} \tag{29}$$

Let

$$N\sigma^2 \rho_0(\alpha) = E\,[X(\psi) \cdot X(\psi + \alpha)] \tag{30}$$

wnere E denotes statistical average. For noise alone, $E\,[X(\psi) . X(\psi + \alpha)]$ $E\,[X(0) . X(\alpha)]$, because of stationarity.

Thus for noise alone

$$N\sigma^2 \rho_0(\alpha) = E\left[\sum_{n=1}^{N} r_n \cos\theta_n \cdot \sum_{n=1}^{N} r_n \cos(\theta_n - n\alpha)\right]$$

but

$$E(r_n \cos\theta_n \cdot r_m \cos(\theta_m - m\alpha)) = 0 \qquad \text{for } m \neq n$$

$$\therefore N\sigma^2 \rho_0(\alpha) = \sum_{n=1}^{N} E(r_n^2 \cos\theta_n \cdot \cos(\theta_n - n\alpha))$$

$$= \sum_{n=1}^{N} E\left(\frac{r_n^2}{2}\,[\cos(2\theta_n - n\alpha) + \cos n\alpha]\right)$$

$$= \sum_{n=1}^{N} E\left(\frac{r_n^2}{2} \cos n\alpha\right) = \sigma^2 \sum_{n=1}^{N} \cos n\alpha$$

$$= \sigma^2 \left[\frac{\sin\left(N + \frac{1}{2}\right)\alpha - \sin\frac{\alpha}{2}}{2 \sin\frac{\alpha}{2}}\right] \tag{31a}$$

$$= \sigma^2 \; \frac{\cos\left(\frac{N+1}{2}\,\alpha\right) \sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}} \tag{31b}$$

Similarly, let

$$N\sigma^2 \lambda_0(\alpha) = E[X(\psi) \cdot Y(\psi + \alpha)] = E[X(0) \cdot Y(\alpha)] \tag{32}$$

$$= \sum_{n=1}^{N} E[r_n^{\,2} \cos\theta_n \cdot \sin(\theta_n - n\alpha)]$$

$$= \sum_{n=1}^{N} E\left[\frac{r_n^{\,2}}{2} (\sin[2\theta_n - n\alpha] - \sin n\alpha)\right]$$

$$= -\sum_{n=1}^{N} E\left(\frac{r_n^{\,2}}{2} \sin n\alpha\right) = -\sigma^2 \sum_{n=1}^{N} \sin n\alpha$$

$$= \sigma^2 \left[\frac{\cos(N+\frac{1}{2})\alpha - \cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}\right] \tag{33a}$$

$$= -\sigma^2 \frac{\sin\left(\frac{N+1}{2}\alpha\right) \sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}} \tag{33b}$$

$\rho_0(\alpha)$ and $\lambda_0(\alpha)$ are normalized covariance functions for the bivariate gaussian distribution, $\rho_0(0) = 1$ and $\lambda_0(0) = 0$.

Let

$$\rho_0 + j\lambda_0 = K_0 e^{j\phi_0} \tag{34}$$

$$K_0 = \sqrt{\rho_0^{\,2} + \lambda_0^{\,2}}, \quad \phi_0 = \tan^{-1}\frac{\lambda_0}{\rho_0}$$

$$K_0(\alpha) = \frac{1}{N}\sqrt{\frac{\cos^2\frac{N+1}{2}\alpha \, \sin^2\frac{N\alpha}{2} + \sin^2\frac{N+1}{2}\alpha \, \sin^2\frac{N\alpha}{2}}{\sin^2\frac{\alpha}{2}}} = \frac{\sin\frac{N\alpha}{2}}{N\sin\frac{\alpha}{2}} \tag{35}$$

$$\phi_0(\alpha) = \tan^{-1}\left(-\tan\left(\frac{N+1}{2}\right)\alpha\right) = -\frac{N+1}{2}\alpha \tag{36}$$

The notation ρ_0, λ_0, K_0 and ϕ_0 is after Middleton.[(4)]

With these statistics for the noise, it is possible to calculate the probability that noise will exceed a given threshold is the interval from 0 to 2π.

Let us consider the average number of crossings with positive slope of a given level R_0, by the process $R(t)$, during some long interval T. This will tell us the average number of noise spikes $\overline{n}_+$ which exceed a given threshold in T. The probability that R is between R_0 and $R_0 + dR$, is

$$p(R_0)\, dR = \frac{\overline{n}\,\overline{\Delta t}}{T} \tag{37}$$

where $\overline{n}$ is the average number of crossings of the level, (of both slopes), and $\overline{\Delta t}$ is the average time required for a crossing. But

$$\Delta t = \frac{dR}{|\dot{R}|} \tag{38}$$

The average value of Δt is given by,

$$\overline{\Delta t} = \frac{dR}{\overline{|\dot{R}|}} = \frac{dR}{\int_{-\infty}^{\infty} |\dot{R}|\, p(\dot{R}/R_0)\, d\dot{R}}. \tag{39}$$

where $p(\dot{R}/R_0)$ is the probability density distribution for $\dot{R}$ given $R = R_0$. Solving for $\overline{n}$

$$\overline{n} = \frac{T\, p(R_0) dR}{\overline{\Delta t}} = T \int_{-\infty}^{\infty} |\dot{R}|\, p(R_0)\, p(\dot{R}/R_0) d\dot{R} = T \int_{-\infty}^{\infty} |\dot{R}|\, p(\dot{R}, R_0) d\dot{R} \tag{40}$$

This is the total number of crossings with both positive and negative slope. The number of crossings with positive slope is

$$\overline{n}_+ = \frac{T}{2} \int_{-\infty}^{\infty} |\dot{R}|\, p(\dot{R}, R_0) d\dot{R} \tag{41}$$

which if $p(\dot{R}/R_0)$ is an even function, is

$$\overline{n}_+ = T \int_0^{\infty} \dot{R}\, p(\dot{R}, R_0) d\dot{R} \tag{42}$$

In our particular ease, R is Rayleigh distributed, and we are working over an angle ψ rather than t. It is shown in Appendix B and elsewhere[4] that,

$$p(\dot{R}, R_0) = \frac{R_0}{\sigma^2} e^{-\frac{R_0^2}{2\sigma^2}} \cdot \frac{e^{-\frac{\dot{R}^2}{-2\sigma^2 K_0''(0)}}}{\sqrt{-2\pi\sigma^2 K_0''(0)}} \tag{43}$$

Thus

$$\overline{n}_+ = \frac{T R_0 e^{-\frac{R_0^2}{2\sigma^2}}}{\sigma^2} \int_0^{\infty} \frac{\dot{R}\, e^{-\frac{\dot{R}^2}{2\sigma^2 K_0''(0)}}}{\sqrt{-2\pi\, \sigma^2 K_0''(0)}}\, d\dot{R}$$

$$= \frac{T R_0 \sqrt{-\sigma^2 K_0''(0)}}{\sigma^2 \sqrt{2\pi}}\, e^{-\frac{R_0^2}{2\sigma^2}} \tag{44}$$

and the phase is $-\frac{N+1}{2}\alpha$, which are the same as the covariance functions, $k_0(\alpha)$ and $\rho_0(\alpha)$, for the noise. The amplitude at $\alpha = 0$ is $NA = A'$. The probability distribution for signal plus noise at $\alpha = 0$ is the familiar probability density distribution for the envelope of sine wave in gaussian noise.[(2), (4)]

$$p(R) \frac{Re^{-\frac{R^2+A'^2}{2N\sigma^2}} I_0\left(\frac{A'R}{N\sigma^2}\right)}{N\sigma^2} \tag{49}$$

But since we have taken the noise power $N\sigma^2 = 1$,

$$p(R) = Re^{-\frac{R^2+A'^2}{2}} I_0(A'R) \tag{50}$$

The peak of the signal, however, will not necessarily coincide with the peak of signal plus noise, although the the two should be very close, since A' is of the order of five. We shall calculate the small increase in R peak due to the fact that the slope of R may not be zero at $\alpha = 0$.

The signal is

$$\frac{A'}{N} \frac{\sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}} \tag{51}$$

This has a main peak at $\alpha = 0$, (or 2π), of A', and many much smaller peaks at which there is a very small probability that signal plus noise will exceed the threshold R_0. Thus, we will consider signal plus noise only in the vicinity of the main peak. For small α we can approximate the signal by

$$S \simeq A' \frac{\sin\frac{N\alpha}{2}}{\frac{N\alpha}{2}} \tag{52}$$

Expanding into series this is

$$S \simeq A'\left[1 - \frac{\left(\frac{N\alpha}{2}\right)}{6} + \frac{\left(\frac{N\alpha}{2}\right)^4}{120} \cdots\right] \tag{53}$$

and

$$\frac{ds}{d\alpha} \simeq A'\left[-\frac{\left(\frac{N\alpha}{2}\right)N}{6} + \frac{\left(\frac{N\alpha}{2}\right)^3 N}{60} \cdots\right] \tag{54}$$

In our particular case we are interested in the probability of noise crossing the threshold over an interval of ψ, of 2π. The probability that noise will cross twice in the interval can be ignored, since we are interested in thresholds high enough that $P_n = 10^{-6}$. For our process the variance is $N\sigma^2$ not σ^2. The expression for $K_0(\alpha)$ is given in (35). If this is differentiated twice and α set to zero we get

$$K_0''(0) = -\frac{N^2-1}{12} \tag{45}$$

Thus, our formula for P_n is

$$P_n = \frac{\sqrt{\frac{N\sigma^2(N^2-1)}{12}}}{N\sigma^2\sqrt{2\pi}} R_0 e^{\frac{-R_0^2}{2N\sigma^2}}$$

$$= \frac{\sqrt{\frac{\pi N\sigma^2(N^2-1)}{6}}}{N\sigma^2} R_0 e^{\frac{-R_0^2}{2N\sigma^2}} \tag{46}$$

For ease of calculation, the noise power out of the summer shall be taken as one, i.e., $N\sigma^2 = 1$. Then

$$P_n = \sqrt{\frac{\pi(N^2-1)}{6}}\, R_0 e^{-\frac{R_0^2}{2}} \tag{47}$$

For a specific N and P_n, R_0 can be solved for.

With the bias level now determined, it is desired to find the signal strength required to give a certain probability of detection, P_d. The signal from the n^{th} return is $A \cos\left[\omega_c t + \delta + n\phi\right]$. Again we may assume $\delta = 0$ without loss of generality. At the output of the summer, the signal has an x component of

$$A \sum_{n=1}^{N} \cos n(\phi-\psi)$$

and a y component of

$$-A \sum_{n=1}^{N} \sin n(\phi-\psi)$$

Let $(\phi - \psi) = \alpha$. Then the envelope of the signal is

$$A \frac{\sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}} \tag{48}$$

We will use an approximation for S which uses only the first two terms of the expansion, thus approximating

$$A' \frac{\sin \frac{N\alpha}{2}}{N \sin \frac{\alpha}{2}}$$

in the region of small $N\alpha/2$ by a parabola. (This will give an estimate for $\bar{\Delta} R$, the average increase in the peak, smaller than it actually is). (See Fig. 4).

$$\therefore \quad S \simeq A' \left[1 - \frac{N^2\alpha^2}{24}\right], \quad \frac{ds}{d\alpha} \simeq - \frac{A N^2 \alpha}{12} \tag{55}$$

If the signal plus noise has a slope of $\dot{R}$ at $\alpha = 0$, the peak of signal plus noise will occur at the point where

$$\dot{R} = - \frac{ds}{d\alpha} = \frac{AN^2\alpha}{12} \tag{56}$$

if $N\alpha$ is small. The amplitude of the peak above the amplitude at $\alpha = 0$ is

$$\Delta R = \alpha \left[\frac{A'N^2\alpha}{12}\right] - \frac{A'N^2\alpha^2}{24} = \frac{A'\alpha^2 N^2}{24} \tag{57}$$

substituting for α from (56)

$$\Delta R = \frac{A'\alpha^2 N^2}{24} = \frac{A'N^2}{24} \left[\frac{12\dot{R}}{A N^2}\right]^2 = \frac{6\dot{R}^2}{A'N^2} \tag{58}$$

The probability distribution for the slope of signal plus noise when the signal is constant, $(ds/d\alpha = 0)$, and the phase modulation of the signal is the same as that for noise, is given in Middleton[4] as

$$p(\dot{R}/R_0) = \frac{e^{-\dot{R}^2/-2N\sigma^2 K_0''(0)}}{\sqrt{-2\pi N\sigma^2 K_0''(0)}} \tag{59}$$

which is independent of signal amplitude.

$$\therefore \; E(\Delta R) = \frac{\sigma}{A'N} = \frac{(N^2-1)}{12} \simeq \frac{1}{2A'} \quad \text{for } N > 0 \tag{60}$$

We have assumed $N\alpha^2 = 1$ and have solved for $K_0''(0) = -(N^2-1)/12$

$$\therefore\ E(\Delta R) = \frac{6}{A'N}\ \frac{(N^2-1)}{12} \simeq \frac{1}{2A'} \text{ for } N > 10 \tag{61}$$

The means of computing A' with a given P_d is then as follows. The probability of detection with a given signal A' is

$$Pd = \int_{R_0}^{\infty} R\, e^{-\frac{R^2 + A'^2}{2}} I_0(A'R)\, dR \tag{62}$$

This integral is tabulated in Marcum[1] as the Incomplete Toronto Function.

$$1 - T_{\frac{R_0}{\sqrt{2}}}\left(1, 0, \frac{A'}{\sqrt{2}}\right) = \int_{R_0}^{\infty} R e^{-\frac{R^2+A'^2}{2}} I_0(A'R)\, dR \tag{63}$$

For a given value of P_d and R_0, A' can then be found. $1/N(A' - \frac{1}{2A'})$ is the amplitude of the signal to the integrator that is required to give the desired probability of detection. Figs. 5 and 6 show a comparison between the integration loss for the coherent detector just discussed, and the ideal non-coherent detector discussed by Marcum[1]. The integration loss is defined as the ratio in d.b. of the total input signal power for a given P_d and P_n, to the total power that would be required in one pulse for the same P_d and P_n. The integrator derived when the doppler velocity of the target was known, had zero d.b. integration loss for all values of N, P_d and P_n.

Chapter 5

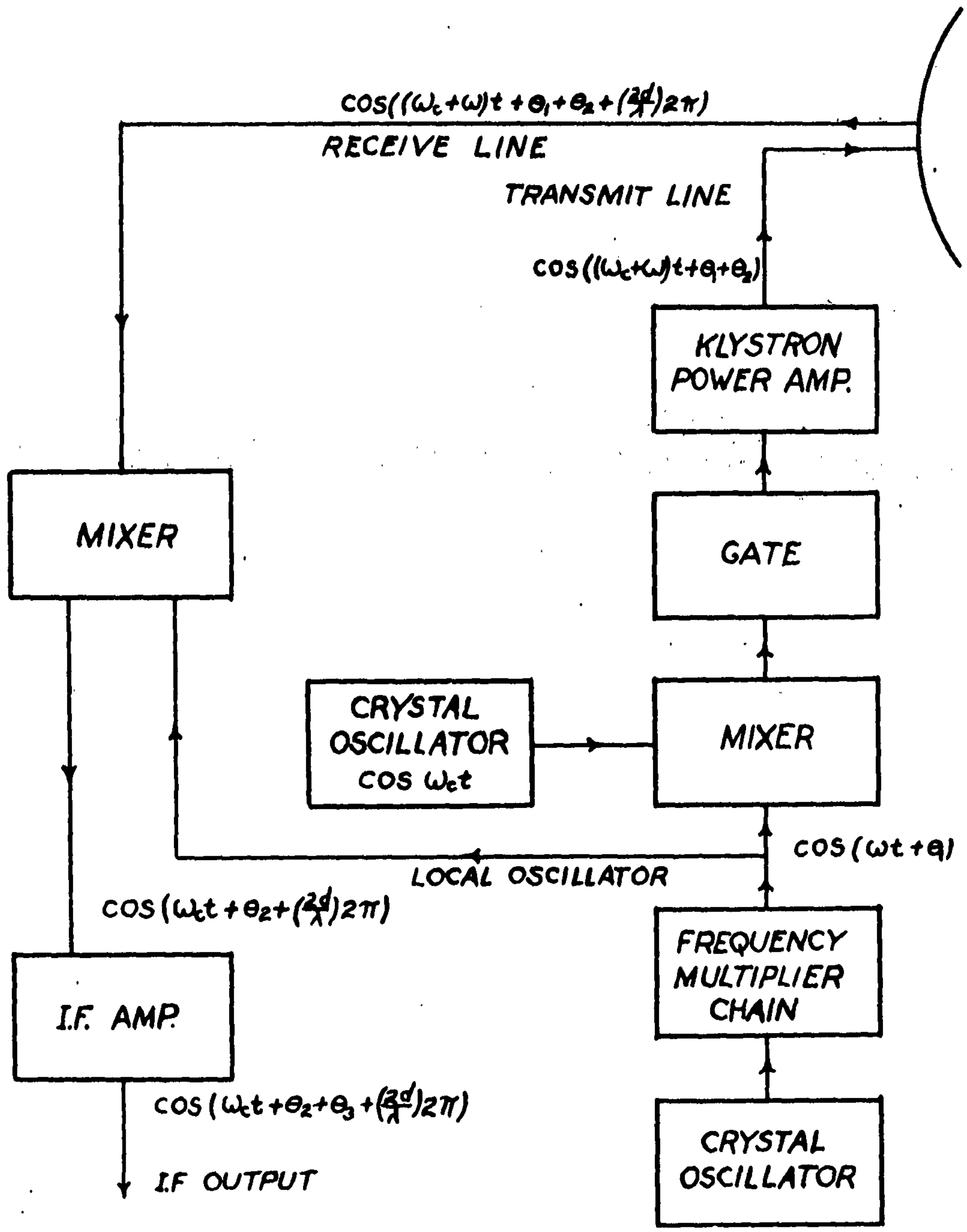
CONCLUSIONS

The results plotted on the graphs of Fig. 5 and 6 show the superiority of coherent over noncoherent integration. When the number of pulses integrated is greater than 30, the improvement is significant. At 100 pulses integrated, it is about 3.5 d.b. Thus, a transmitter of one half the power would give slightly better performance with a coherent integrator, than a full power transmitter would give with a noncoherent integrator, when the antenna beamwidth contains 100 target returns. Since transmitter power is expensive, use of coherent integration on such a system might be economically feasible.

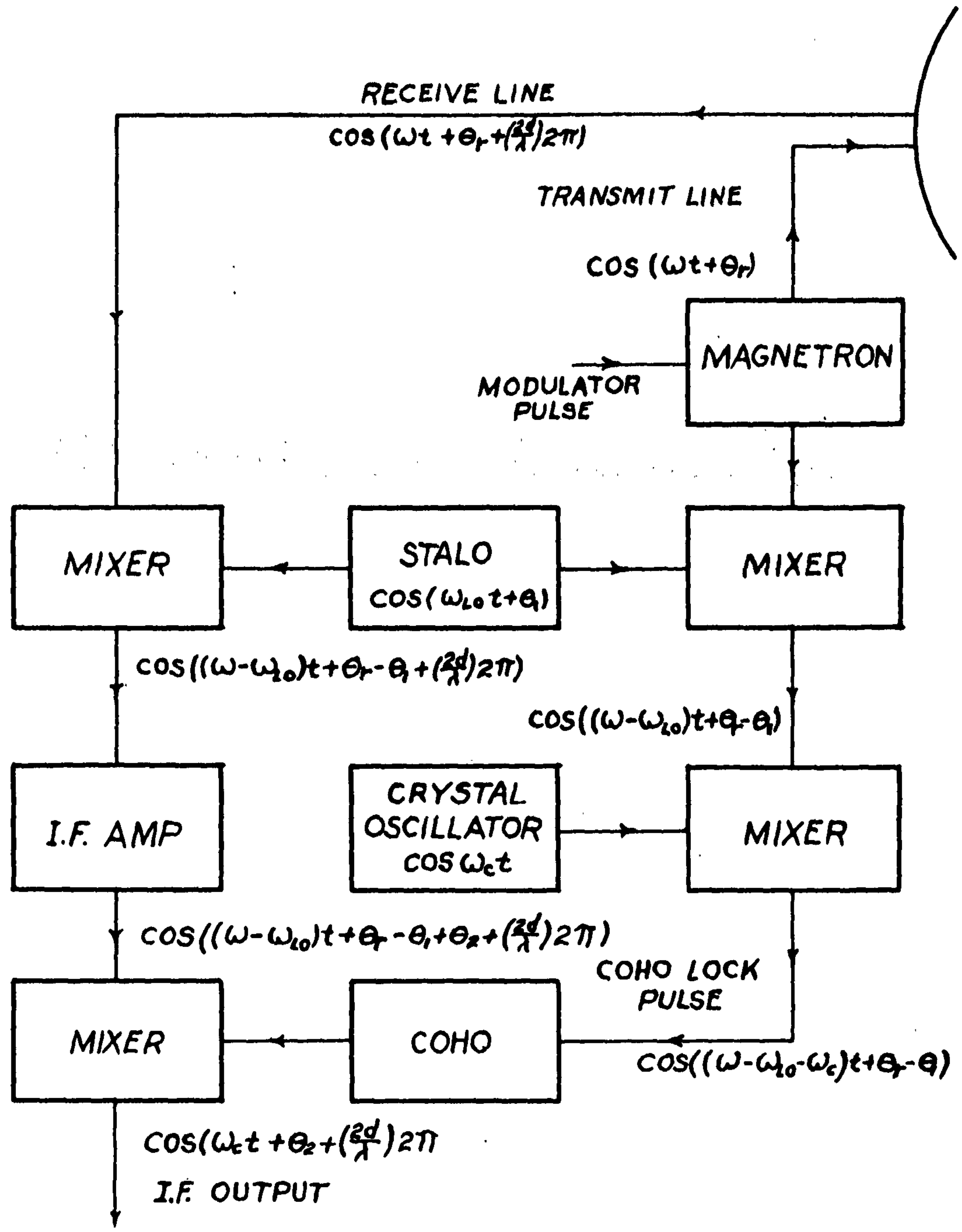
It should be noted that the coherent integrator performs better when $P_n = 10^{-10}$ rather than 10^{-6}. This is probabily because at $P_n = 10^{-10}$, the threshold is higher, and the integrator more closely approximates the Neyman Pearson test. That is,

$$\int_0^{2\pi} I_0\left(\frac{AR(\psi)}{\sigma^2}\right)d\psi$$

is more dependent on the peaks for larger values of $AR(\psi)/\sigma^2$.

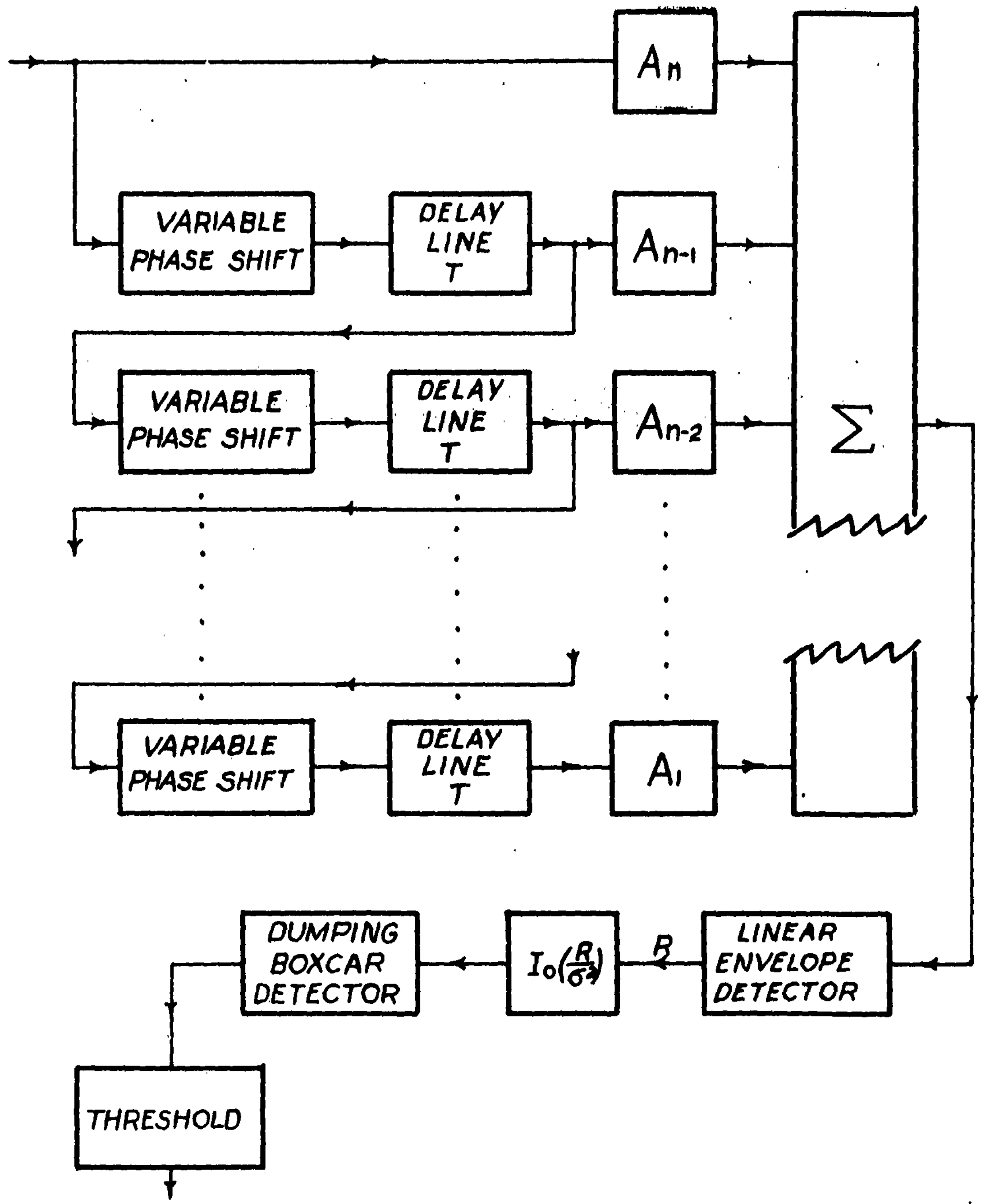


Crystal Controlled Radar
Fig 1



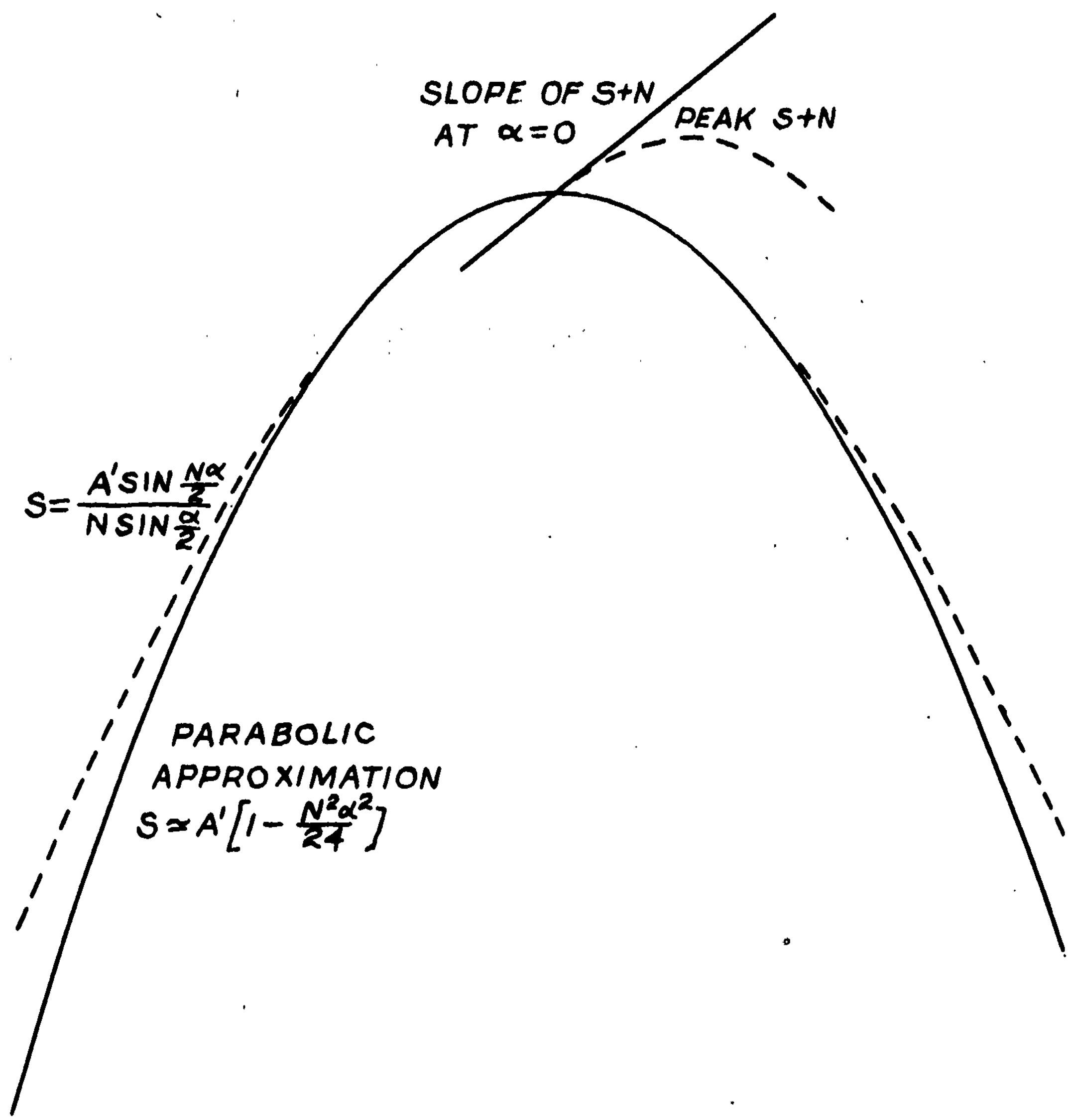
Magnetron Radar

Fig 2

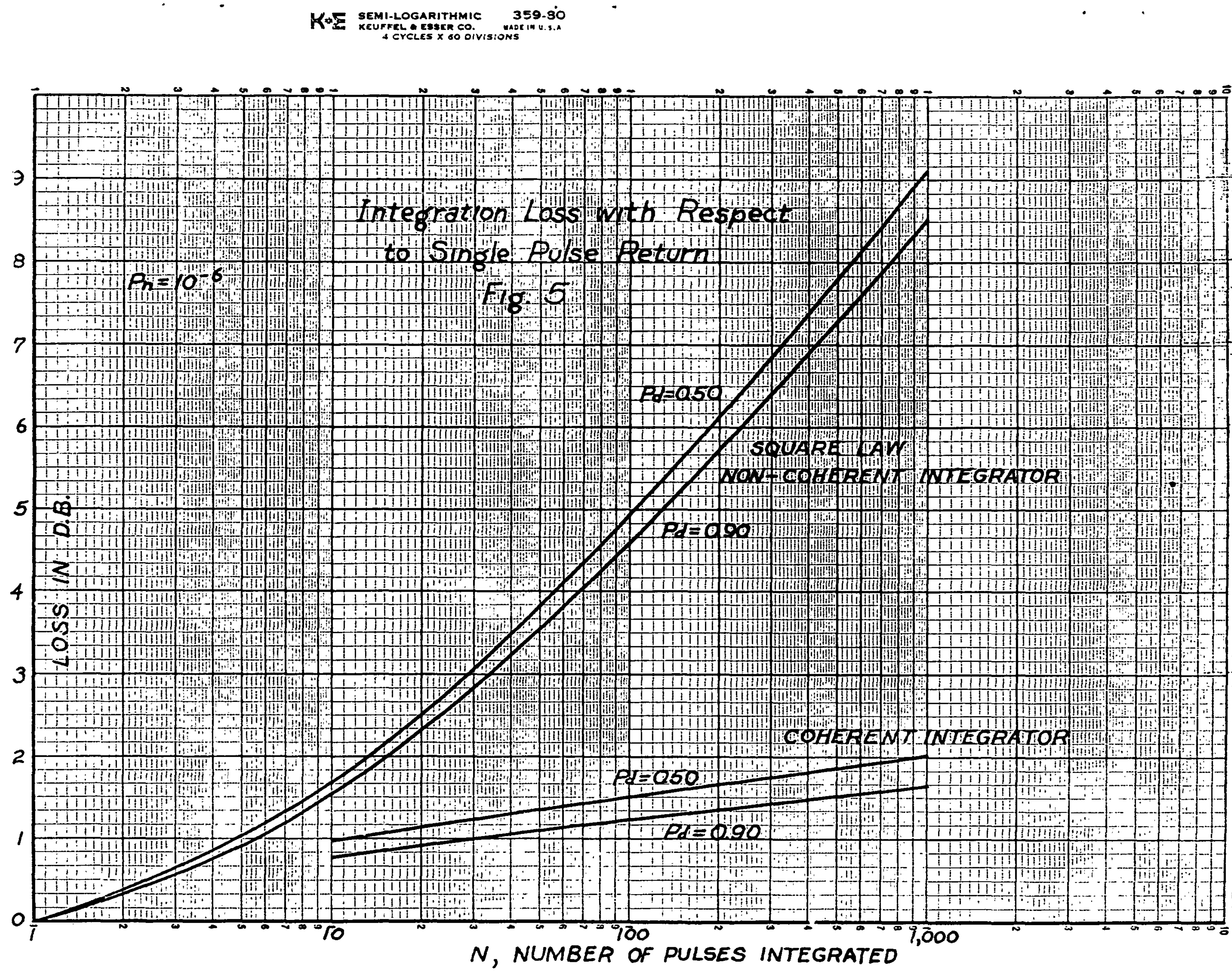


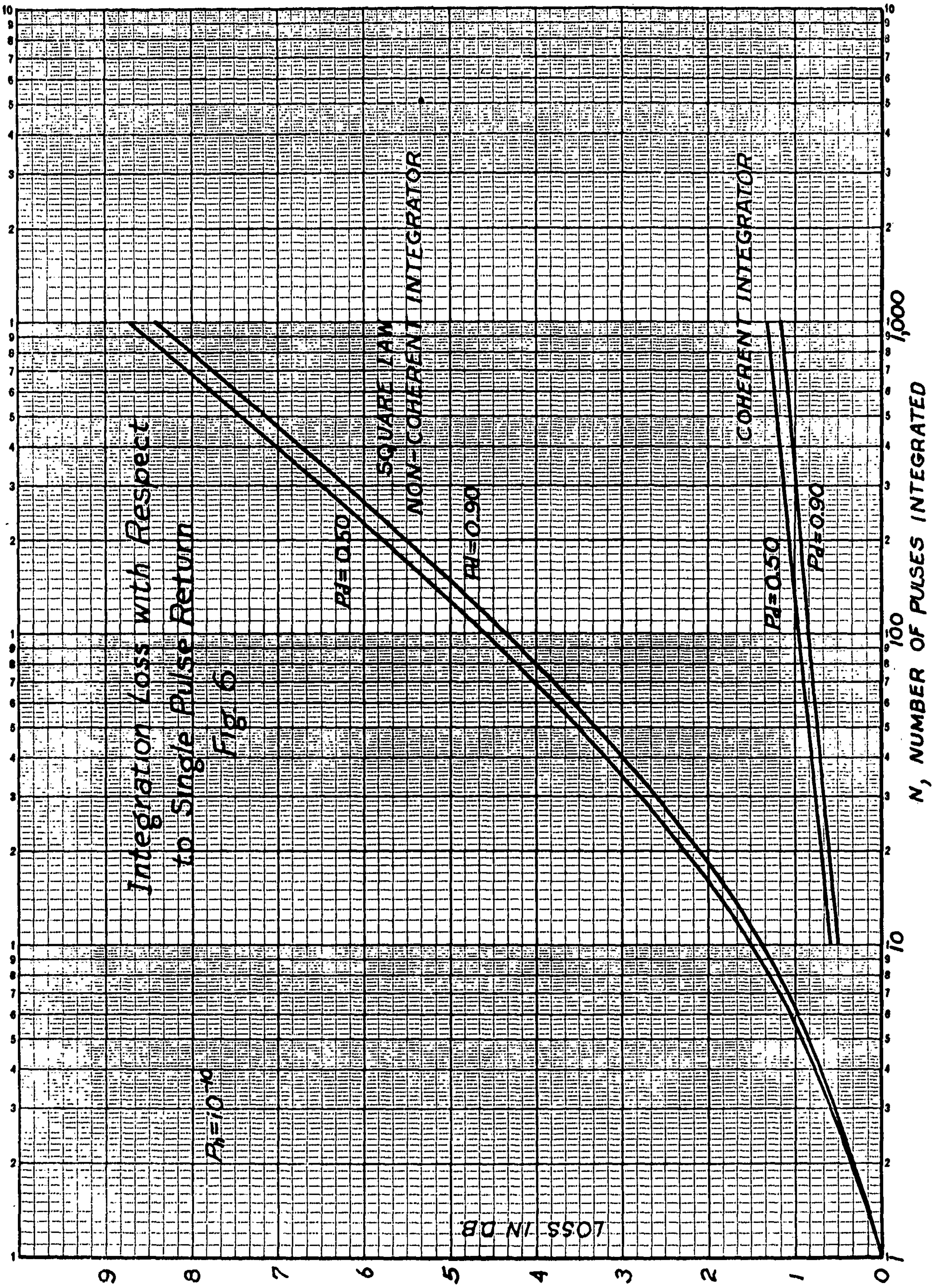
Neyman-Pearson Coherent Detector

Fig 3



Estimation of Peak of Signal + Noise
Fig 4


K+E SEMI-LOGARITHMIC 359-80
KEUFFEL & ESSER CO. MADE IN U.S.A
4 CYCLES X 60 DIVISIONS
Integration Loss with Respect
to Single Pulse Return
Fig. 5
$P_n = 10^{-6}$
$P_d = 0.50$
SQUARE LAW
NON-COHERENT INTEGRATOR
$P_d = 0.90$
COHERENT INTEGRATOR
$P_d = 0.50$
$P_d = 0.90$
LOSS IN D.B.
0
1
2
3
4
5
6
7
8
9
1
10
100
1,000
N, NUMBER OF PULSES INTEGRATED



Integration Loss with Respect to Single Pulse Return

Fig 6

Appendix A

NEYMAN PEARSON TEST FOR COMPOSITE HYPOTHESIS vs SIMPLE HYPOTHESIS

Let $y = \{y_1, y_2, \ldots, y_n\}$ be point in n-space representing set of observed parameters. $p_0(y)$: prob. d.d. of simple hypothesis H_0

$p_{1_{\alpha_1, \alpha_2, \ldots \alpha_m}}(y)$: prob. d.d. of composite hypothesis H_1 as function of parameters $\alpha_1, \alpha_2, \ldots \alpha_m$.

Then

$$p_1(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p_{1_{\alpha_1, \ldots \alpha_m}}(y)\, p(\alpha_1)p(\alpha_2)\ldots p(\alpha_m)d\alpha_1\ldots d\alpha_m$$

is prob. d.d. of hypothesis H_1.

Y is the space of observations

Y_0 is the set of y's such that H_0 is chosen

Y_1 is set of y's such that H_1 is chosen

$$Y = Y_0 \cup Y_1, \quad Y_0 \cap Y_1 = 0$$

$P_0(Y_1)$ is level of test (prob. of accepting H_1 when it is false)

$$P_0(Y_1) = \int_{Y_1} p_0(y)dy.$$

$P_1(Y_1)$ is power of test (prob. of accepting H_1 when it is true)

$$P_1(Y_1) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty} P_{1_{\alpha_1, \ldots \alpha_m}}(y)\, dy\, p(\alpha_1)p(\alpha_2)\ldots p(\alpha_m)d\alpha_1 d\alpha_2\ldots d\alpha_m$$

$$\therefore P_1(Y_1) = \int_{Y_1} p_1(y)dy$$

Choose Y_1 so that in Y_1, $p_1(y)/p_0(y) \geq \eta$.

In Y_0, $p_1(y)/p_0(y) < \eta$.

Let $\alpha = P_0(Y_1)$ be level of test.

Let T_1 be set of y's such that $P_0(T_1) \leq \alpha$.

Then it can be shown that $P_1(Y_1) \geq P_1(T_1)$; i. e., the likelihood ratio method of choosing Y_1 gives a test of maximum power for a given level.

Proof: Let $A = T_1 \cap Y_1$

$$P_1(Y_1 - A) = \int_{Y_1 - A} p_1(y)dy \geq \eta \int_{Y_1 - A} p_0(y)dy = \eta\, P_0(Y_1 - A)$$

because $(Y_1 - A) \subset Y_1$

$$P_0(Y_1 - A) = P_0(Y_1) - P_0(A) = \alpha - P_0(A)$$

$$\therefore\; P_1(Y_1) = P_1(Y_1 - A) + P_1(A) \geq \eta\, \alpha - \eta\, P_0(A) + P_1(A)$$

$$P_1(T_1 - A) = \int_{T_1 - A} p_1(y)dy \leq \int_{T_1 - A} p_0(y)dy = \eta\, P_0(T_1 - A)$$

because $T_1 - A \subset Y_0$

$$P_0(T_1 - A) = P_0(T_1) - P_0(A) \leq \alpha - P_0(A)$$

$$\therefore\; P_1(T_1) = P_1(T_1 - A) + P_1(A) \leq \eta\, \alpha - \eta\, P_0(A) + P_1(A)$$

$$\therefore\; P_1(Y_1) \geq P_1(T_1)$$

Appendix B

PROBABILITY DENSITY DISTRIBUTION FOR ENVELOPE AND SLOPE OF ENVELOPE OF GAUSSIAN PROCESS

The slope of a sample function of a process is given by

$$\dot{R} = \lim_{\Delta\tau \to 0} \frac{R_2 - R_1}{\Delta\tau}$$

where R_1 and R_2 are the values of the sample function at times $\Delta\tau$ apart. The probability density distribution for $\dot{R}$ given R_1 is then

$$P(\dot{R}/R_1) = \lim_{\Delta\tau \to 0} P\left(\frac{R_2 - R_1}{\Delta\tau}\right)$$

The joint p. d. d. for R_1 and R_2 is well known(2),(4) for a Rayleigh Process, and is

$$P(R_1, R_2) = \frac{R_1 R_2}{\sigma^4\left(1 - K_0^2(\tau)\right)} e^{-\frac{R_1^2 + R_2^2}{2\sigma^2(1 - K_0^2(\tau))}} I_0\left[\frac{K_0(\tau) R_1 R_2}{\sigma^2\left(1 - K_0^2(\tau)\right)}\right]$$

For small τ we can expand $K_0(\tau)$ into a MacLauren series

$$K_0(\tau) = K_0(0) + K_0'(0)\tau + \frac{1}{2} K_0''(0)\tau^2 + \cdots$$

But $K_0(0) = 1$ and since $K_0(\tau)$ is a maximum at zero, $K_0'(0) = 0$

$$\therefore \quad K_0(\tau) \simeq 1 + \frac{1}{2} K_0''(0)\tau^2 \quad \text{for small } \tau$$

$$K_0^2(\tau) \simeq 1 + K_0''(0)\tau^2 \qquad \text{for small } \tau$$

Thus for small τ

$$p(R_1, R_2) = \frac{R_1 R_2}{-\sigma^4 K_0''(0)\tau^2} e^{-\frac{R_1^2 + R_2^2}{-2\sigma^2 K_0''(0)\tau^2}} I_0\left[\frac{R_1 R_2}{\sigma^2}\left(\frac{1}{-K_0''(0)\tau^2} - \frac{1}{2}\right)\right]$$

But $I_0(x) \simeq \frac{e^x}{\sqrt{2\pi x}}$ for large x

$$\therefore \quad p(R_1, R_2) \simeq \frac{\sqrt{R_1 R_2}\; e^{-\frac{(R_2 - R_1)^2}{-2\sigma^2 K_0''(0)\tau^2}}\; e^{-\frac{R_1 R_2}{2\sigma^2}}}{\sigma^2\sqrt{-2\pi\sigma^2 K_0''(0)\tau^2}}$$

As $\tau \to 0$ R_1 must approach R_2 so we may say

$$p(R_1, R_2) \simeq \frac{R_1}{\sigma^2} \; e^{-\frac{R_1^2}{2\sigma^2}} \; \frac{e^{-\frac{(R_2-R_1)^2}{-2\sigma^2 K_0''(0)\tau^2}}}{\sqrt{-2\pi\sigma^2 K_0''(0)\,\tau^2}}$$

By a simple transformation

$$p\left(R_1, \frac{R_2-R_1}{\tau}\right)_{\tau\to 0} = p(R_1, \dot{R}) = \frac{R_1 e^{-\frac{R_1^2}{2\sigma^2}}}{\sigma^2} \cdot \frac{e^{-\frac{\dot{R}^2}{-2\sigma^2 K_0''(0)}}}{\sqrt{-2\pi\,\sigma^2 K_0''(0)}}$$

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GLOSSARY OF PRINCIPAL SYMBOLS USED IN BULK OF REPORT

A, A'	= amplitudes of I. F. signals
A_n	= amplitude of I. F. from n^{th} return
α	= an angle $(\phi-\psi)$
C	= a constant
d	= distance of target from radar
$E(x)$	= statistical average of x, sometimes written $\overline{x}$
f_c	= center frequency of I. F.
δ	= constant phase angle
H_0, H_1	= null and alternate hypotheses
I_0	= modified Bessel function of first kind and zero order
K, K', K''	= constants, numbers
K_0	= $\sqrt{\rho_0^2+\lambda_0^2}$ = normalized covariance function
λ	= wavelength of transmitted R. F.
$\lambda_0(\alpha)$	= $\frac{1}{N\sigma^2} E(X(\psi)\cdot Y(\psi+\alpha))$ = normalized covariance function
N	= number of pulse returns from a target
n	= integer; number of threshold crossings
n_+	= number of positive threshold crossings
ω	= radian frequency
ω_c	= $2\pi f_c$ = radian frequency of center of I. F.
P_n	= probability of false alarm
P_d	= probability of detection
$p(x)$	= probability density distribution of x
$p_N(\{r_n, \theta_n\})$	= p. d. d. of sequence of r_n's and θ_n's due to noise
$p_{S+N}(\{r_n, \theta_n\})$	= p. d. d. of sequence of r_n's and θ_n's due to signal plus noise
$p_{\psi,\delta}S+N(\ ), p_{\psi}S+N(\ )$	= p. d. d. dependent on ψ and δ, and ψ respectively
ϕ_0	= $\tan^{-1}\frac{\lambda_0}{\rho_0}$ = angular correlation function
ϕ	= an angle
ψ	= precision angle of I. F. returns
$R(\psi)$	= envelope amplitude = $\sqrt{X^2(\psi) + Y^2(\psi)}$
R_0	= threshold level
r_n	= observed amplitude of I. F. from n^{th} return

$\rho_0(\alpha)$	$= \frac{1}{N\sigma^2} E(X(\psi)\, X(\psi+\alpha))$ = normalized covariance function
S	= signal
σ^2	= variance, or power of I.F. noise
T	= interpulse period; period of time
τ	= pulse length; time interval
t	= time
θ_n	= observed phase of I.F. from n^{th} return
V(t)	= voltage of process
v	= radial velocity of target
x(t)	= cosine component of I.F. gaussian noise
$X(\psi)$	$= \sum_{n=1}^{N} r_n \cos(\theta_n - n\psi)$
y(t)	= sine component of I.F. gaussian noise
$Y(\psi)$	$= \sum_{n=1}^{N} r_n \sin(\theta_n - n\psi)$

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